MET O 11 TECHNICAL NOTE NO 73

Matrix multiplication by diagonals

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1. Introduction

Vector and array processors will admit of new algorithmic techniques. The Appendix to this note consists of a paper by N.K. Madsen, G.H.Rodrigue, and J.I.Karush in which they establish three "Fortran Theorems" which they claim have relevance to these new processors.

It is not the purpose of this note to assess the validity of their claims as this is a matter for a programmer. The purpose of this note is to relate their Fortran ideas to standard matrix algebra. It seems desirable as a general principle that, where possible, these new types of algorithm should be related to standard matrix algebra in order to facilitate the study of further possibilities and applications.

N.K.MADSEN, G.H.RODRIGUE, and J.I.KARUSH:- "Matrix multiplication by diagonals on a vector/parallel processor" - Information Processing Letters, 5, 2, June 1976, pp 41-45.

N.B. This paper has not been published. Permission to quote from it must be obtained from the Assistant Director of the above Meteorological Office branch.

2. The Matrix Algebra

What is usually meant by the matrix multiplication of two matrices A and B is the scalar product, denoted in texts by AB or A•B. Specifically, for a 3 x 3 example if

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{1}$$

then if C= A•B we have

$$C=\begin{pmatrix} (a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}) & (a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}) & (a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33}) \\ (a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}) & (a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}) & (a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33}) \\ (a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}) & (a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}) & (a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33}) \end{pmatrix} \tag{2}$$

the well-known result.

Another less well-known product of two matrices is the Hadamard product. In terms of A and B as given by (1) it is defined by

$$A \square B=\begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{pmatrix} \tag{3}$$

that is, each element of A is multiplied by the corresponding element of B. As an aside from the theory, it is noticeable immediately that this is a convenient type of product if the elements of each matrix are strung out as vectors in a computer store.

Suppose now that instead of forming $A \square B$ we form $A \square \tilde{B}$, the Hadamard product of A with the transpose of B. The result is

$$A \square \tilde{B}=\begin{pmatrix} a_{11}b_{11} & a_{12}b_{21} & a_{13}b_{31} \\ a_{21}b_{12} & a_{22}b_{22} & a_{23}b_{32} \\ a_{31}b_{13} & a_{32}b_{23} & a_{33}b_{33} \end{pmatrix} \tag{4}$$

It is now apparent by inspection of (2) and (4) that the elements on the main diagonal of A•B are the sums of the elements of the rows of $A \square \tilde{B}$. This means that by introducing the column vector of ones

$$\underline{1}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{5}$$

the main diagonal of A•B can be written as a vector $\underline{c}_o$ as

$$\underline{c}_o = (A \square \tilde{B}) \bullet \underline{1} \tag{6}$$

The other diagonals can also be written in terms of Hadamard products by inventing a little special notation. For example, let

$${}_2A \square {}^2\tilde{B}$$ mean - take the bottom two rows of A and the top two rows of $\tilde{B}$ and form the Hadamard product

and let $\underline{c}_{-2}$, $\underline{c}_{-1}$, $\underline{c}_o$, $\underline{c}_1$, $\underline{c}_2$ be the diagonals of A•B as vectors starting at the bottom left corner of (2), then

$$\begin{aligned}
\underline{c}_{-2} &= ({}_1A \square {}^1\tilde{B}) \bullet \underline{1} \\
\underline{c}_{-1} &= ({}_2A \square {}^2\tilde{B}) \bullet \underline{1} \\
\underline{c}_o &= (A \square \tilde{B}) \bullet \underline{1} \\
\underline{c}_1 &= ({}^2A \square {}_2\tilde{B}) \bullet \underline{1} \\
\underline{c}_2 &= ({}^1A \square {}_1\tilde{B}) \bullet \underline{1}
\end{aligned} \tag{7}$$

3. Hadamard Products

The algebra of the Hadamard products is not easy to come by, being scattered through the literature over about seventy years. Most textbooks give only a passing reference to it, if at all. It has been variously called the Hadamard product, the Schur product, and the Ring product, and it has been denoted by $*$, $\circ$, $\square$, $\boxdot$, $\odot$, and $\times$. Lately, opinion has been settling down in favour of the name Hadamard product and either of the two symbols $\circ$ or $*$. I have used $\square$ as a personal preference simply because $\circ$ is much used to denote composition in algebra and $*$ is usually used to denote an undefined product in vector analysis. Some elementary results which may be useful are

(a) If A and B are of unit rank then they can be expressed in terms of vectors as $A = \underline{u}\tilde{\underline{v}}$ and $B = \underline{w}\tilde{\underline{x}}$. In this case

$$A \square B = (\underline{u}\tilde{\underline{v}}) \square (\underline{w}\tilde{\underline{x}}) = (\underline{u} \square \underline{w})\,(\tilde{\underline{v}} \square \tilde{\underline{x}}) \tag{8}$$

(b) With $\underline{1}$ defined as in (5) above $J = \underline{1}\tilde{\underline{1}}$ is a matrix of ones. It plays the role of identity matrix in Hadamard products. Thus

$$A \square J = J \square A = A \tag{9}$$

(c) Hadamard multiplication is commutative (unlike the usual scalar product multiplication). Thus

$$A \square B = B \square A \tag{10}$$

(d) It is distributive

$$(A + B) \square C = A \square C + B \square C \tag{11}$$

(e) $$\mathrm{Tr}(A \bullet B) = \tilde{\underline{1}} \bullet (A \square \tilde{B}) \bullet \underline{1}$$

(f) If D is a diagonal matrix then

$$D\bullet(A\square B) = (D\bullet A)\square B = A\square(D\bullet B) \tag{12}$$

$$(A\square B)\bullet D = A\square(B\bullet D) = (A\bullet D)\square B \tag{13}$$

$$D\bullet(A\square B)\bullet D = (D\bullet A\bullet D)\square B = A\square(D\bullet B\bullet D) = (D\bullet A)\square(B\bullet D) = (A\bullet D)\square(D\bullet B) \tag{14}$$

(g) If A has the form $A = \underset{\sim}{uv}$, let D_u and D_v be diagonal matrices formed $\underline{u}$ and $\underline{v}$. Then

$$A\square B = (\underset{\sim}{uv})\square B = (D_u\bullet J\bullet D_v)\square B = D_u\bullet(J\square B)\bullet D_v = D_u\bullet B\bullet D_v \tag{15}$$

There is a good deal more at a more esoteric level but the matter can rest here pending the development of interest and utilization.

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